Precalculus, Quarter 4, Unit 4.1

Conic Sections—Ellipses and Hyperbolas

Overview

Number of instructional days: 10 (1 day = 45–60 minutes)

Content to be learned

- Derive the standard equations of ellipses and hyperbolas.
- Change between the general equation for a conic section and standard form.
- Graph elliptic and hyperbolic functions.
- Create the equations of elliptic and hyperbolic functions from their graphs.
- Solve word problems in multiple representations involving hyperbolas and ellipses.
- Write the equation of a hyperbola or ellipse given vertices and foci.

Mathematical practices to be integrated

Model with mathematics.

- Show that the sum of the distances from the foci to the curve is constant for any ellipse.
- Show that the difference of the distances from the foci to the curve is constant for any hyperbola.

Look for and make use of structure.

- Compare the graphs to the standard equations.

Look for and express regularity in repeated reasoning.

- Utilize the standard forms of the ellipse and hyperbola to graph transformations.

Essential questions

- How are parabolic, elliptic, hyperbolic, linear, and circular functions related?
- How are the parts of the various elliptic and hyperbolic functions related from their formulas or graphs?
- What are the similarities and differences between hyperbolic and elliptic equations?
- How is the standard form for elliptical and hyperbolic equations derived from the general form?

Written Curriculum

Common Core State Standards for Mathematical Content

Expressing Geometric Properties with Equations	G-GPE

Translate between the geometric description and the equation for a conic section

G-GPE.3 (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Common Core Standards for Mathematical Practice

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Clarifying the Standards

Prior Learning

In third grade, students made picture graphs and scaled bar graphs. In fourth grade, students made a line plot to find the difference in length of figures. In fifth grade, students began graphing points on the coordinate plane to solve real-world and mathematical problems. In grade 6, students displayed numerical data in plots on a number line, including dot plots, histograms, and box plots. In grade 7, students graphed a coordinate plane and observed whether the graph was a straight line through the origin. In grade 8, students estimated the solutions of a system of equations by graphing. In algebra 1, students graphed linear and quadratic functions. In geometry, students graphed conic sections after learning the Pythagorean and distance theorems in relation to coordinate geometry. In algebra 2, students graphed linear and quadratic functions.

Current Learning

Students derive and utilize the standard forms for elliptical and hyperbolic equations. Students also translate between the general and standard form to graph elliptical and hyperbolic equations.

Future Learning

In future science course work, students will use orbital mechanics to calculate motions of bodies in gravitational fields. Students may use hyperbolic geometry to find the optimal lensing for optical instruments such as optometry or commercial optics.

Additional Findings

Students have difficulties converting from general to standard form for elliptical and hyperbolic equations as well recalling the standard form equations. Students also have difficulties graphing the ellipses and hyperbolas in correct standard form.

Precalculus, Quarter 4, Unit 4.2

Cavalieri's Principle

Overview

Number of instructional days: 10 (1 day = 45–60 minutes)

Content to be learned

- Calculate the volume of three-dimensional figures when cross sections are known.
- Determine the cross sections of three-dimensional shapes.
- Show that the volume of a three-dimensional shape is the sum of the areas of all cross sections of that shape.
- Demonstrate that three-dimensional figures with equal sets of cross sections have the same volume.
- Describe or show how the volume formulas are derived from the areas of the bases multiplied through the height.
- Express the proportional relationships of the area in connection with the volumes of three-dimensional shapes and their cross sections.

Mathematical practices to be integrated

Make sense of problems and persevere in solving them.

- Draw cross sections of various three-dimensional figures.
- Use correct notation to describe the concept of infinite cross sections.

Reason abstractly and quantitatively.

- Verify through deductive reasoning that it takes infinite cross section areas to add to the total volume.

Model with mathematics.

- Compute the volumes and areas of three-dimensional shapes and cross sections.

Essential questions

- How do cross sections determine volumes of three-dimensional shapes?
- How does the concept of infinite sets apply to the sum of the areas of cross sections?
- Where do the formulas for volume of three-dimensional figures come from, and how were they derived?

Written Curriculum

Common Core State Standards for Mathematical Content

Geometric Measurement and Dimension	G-GMD

Explain volume formulas and use them to solve problems

G-GMD.2 (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

Common Core Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Clarifying the Standards

Prior Learning

In first grade, students partitioned circles and rectangles into equal shares. In second grade, students partitioned rectangles in rows and columns into squares and counted to find the total. In third grade, students expressed area in terms of a unit fraction of the whole. In fourth grade, students applied the area formula for rectangles in real-world and mathematical problems.

In fifth grade, students recognized volume as an attribute of solid figures and understood concepts of volume measurement. They related volume to the operations of addition and subtraction and solved real-world, mathematical problems involving volume. In sixth grade, students found the volume of right rectangular prisms by packing them with unit cubes, and they showed that the volume is the same as multiplying the sides and using the formulas. Students found the area of right triangles and other triangles, polygons, and quadrilaterals by composing into rectangles or decomposing into triangles or other shapes.

In seventh grade, students solved real-world, mathematical problems involving volume composed of cubes and right prisms. Students understood the formula for the area of a circle. Students found the area of two-dimensional shapes. In eighth grade, students understood the formula for volumes of cones, cylinders, and spheres. In geometry, students found the area of sectors, explained the volume formulas, and used them to solve problems.

In algebra 2, students worked with the concept of infinity in terms of end behavior, domain/range, and asymptotes.

Current Learning

In this unit, students use the concept of infinite cross sections of various three-dimensional figures to derive volume formulas.

Future Learning

In calculus, students will learn integration techniques through the washer and disk methods to find volumes of rotated solids, given their functions. In fields such as engineering and science, students will maximize different variables using these techniques.

Additional Findings

Students have trouble with the concept of infinite sets and their sums. Students have difficulty visualizing cross sections.

Precalculus, Quarter 4, Unit 4.3

Solving Systems of Equations with Matrices and Inverse Matrices

Overview

Number of instructional days: 10 (1 day = 45–60 minutes)

Content to be learned

- Calculate the volume of three-dimensional figures when cross sections are known.
- Determine the cross sections of three-dimensional shapes.
- Show that the volume of a three-dimensional shape is the sum of the areas of all cross sections of that shape.
- Demonstrate that three-dimensional figures with equal sets of cross sections have the same volume.
- Describe or show how the volume formulas are derived from the areas of the bases multiplied through the height.
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Mathematical practices to be integrated

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- Draw cross sections of various three-dimensional figures.
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Reason abstractly and quantitatively.

- Verify through deductive reasoning that it takes infinite cross section areas to add to the total volume.

Model with mathematics.

- Compute the volumes and areas of three-dimensional shapes and cross sections.

Essential questions

- How do cross sections determine volumes of three-dimensional shapes?
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Written Curriculum

Common Core State Standards for Mathematical Content

Geometric Measurement and Dimension	G-GMD

Explain volume formulas and use them to solve problems

G-GMD.2 (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

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